Jan 19, 2014

## The Area Problem

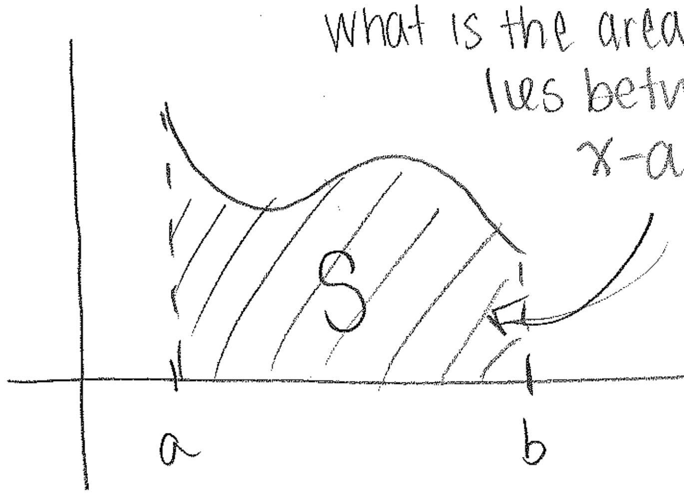
Tangent line problem led up to derivatives

Area Problem leads to "integrals" (which we will ultimately discover are antiderivatives).

What is the area problem?

Given function f(x) defined from a to b:

what is the area of the region that lies between f and the x-axis?



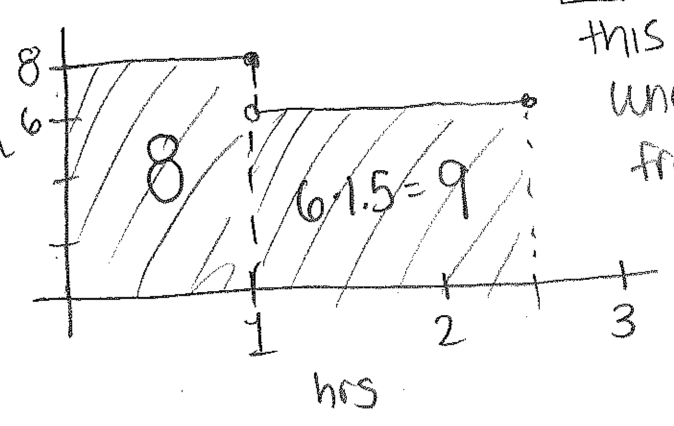

Why do we care?

- Let's say a marathon runner goes 8mph for 1 hour and 6 mph for another 1½ hours.

How far did the runner travel?

$$8 \cdot 1 + 6 \cdot 1.5 = 8 + 9 = \underbrace{17}\ \text{mi.}$$

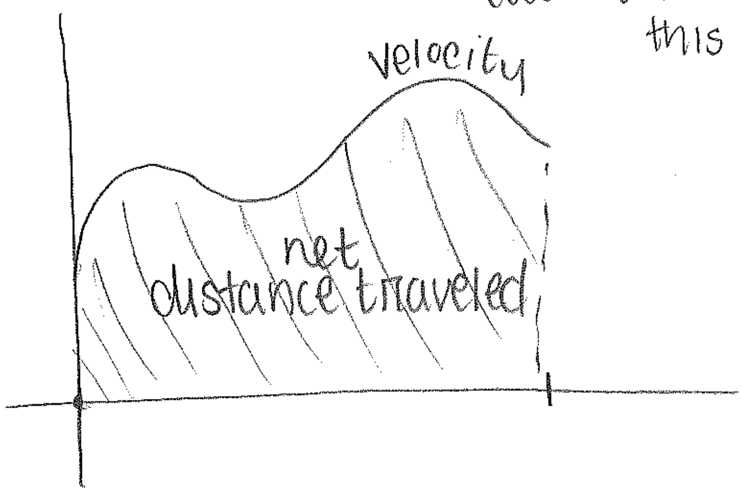
Graph:



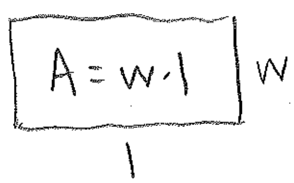

this is the area under the graph from 0 to 1.5

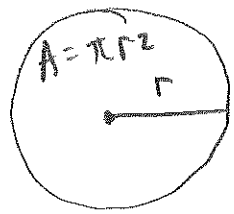
In general: area under velocity curve will be net distance traveled:

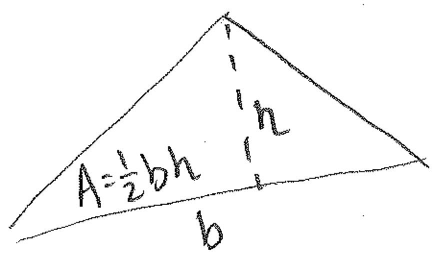
(we'll see more about this later)



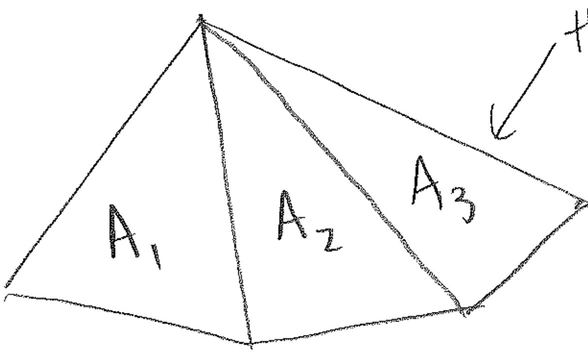

So, <u>area</u>: we know some formulas for area




we also know area is additive




the area of this entire shape is the sum of the area of the parts

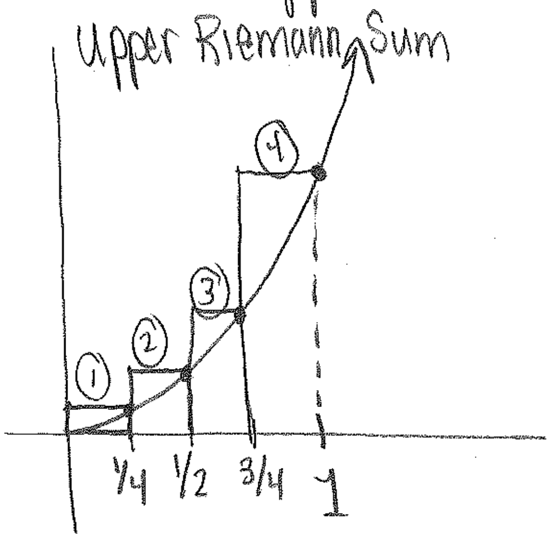
$$\text{Area} = A_1 + A_2 + A_3$$

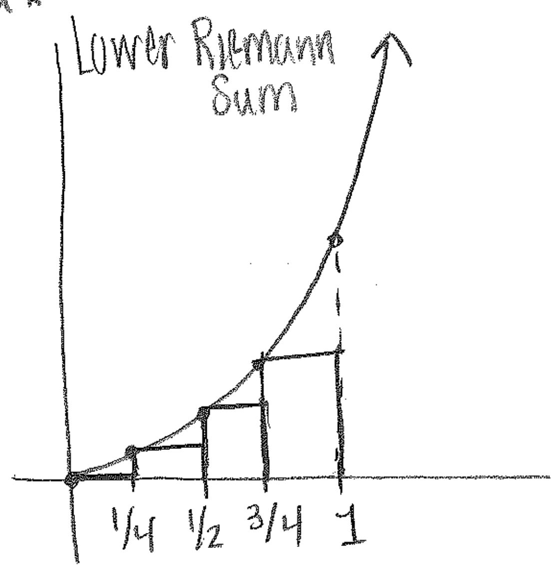
But how do we get the area of S?

Let's start in the same way we did for tangent lines, by approximating.

<u>Ex:</u> $f(x) = x^2$. Approximate area under $f(x)$ from 0 to 1.

*We are really good with areas of rectangles. Let's use these to approximate area*




add up the area of these rectangles

width of rectangles is $1/4$

height is height of function

① $1/4 \cdot f(1/4) = \frac{1}{4} \cdot \frac{1}{16} = \frac{1}{64}$

② $1/4 \cdot f(1/2) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

③ $1/4 \cdot f(3/4) = \frac{1}{4} \cdot \frac{9}{16} = \frac{9}{64}$

④ $1/4 \cdot f(1) = \frac{1}{4}$

① + ② + ③ + ④ $= 15/32$

area w/ these rectangles is

$0^2 \cdot 1/4 + (1/4)^2 \cdot 1/4 + (1/2)^2 \cdot 1/4 + (3/4)^2 \cdot 1/4$
$= 7/32$

So

$7/32 < \text{Area} < 15/32$

If we use more rectangles, we get a better estimate of area.

Estimating w/ Rectangles Like this are called <u>Riemann Sums</u>

To estimate the area under a function $f(x)$ between a and b:

(1) Decide how many rectangles you will use. This # is $n$.

(2) Divide interval $[a,b]$ into $n$ equal intervals,

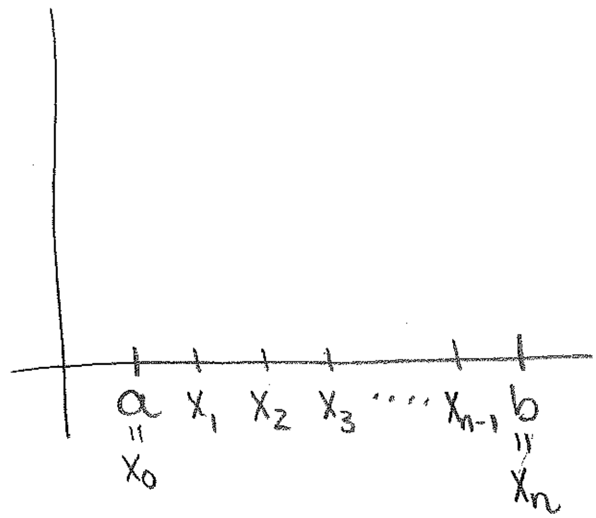
$[x_0, x_1]$

$[x_1, x_2]$

$[x_2, x_3]$

$\vdots$

$[x_{n-1}, x_n]$




The $x_i$'s can be figured out as follows:

Let $\Delta x$ be the width of rectangles we will draw.

Then $\Delta x = \dfrac{b-a}{n}$

So $x_0 = a$

$x_1 = a + \Delta x$

$x_2 = a + 2\Delta x$

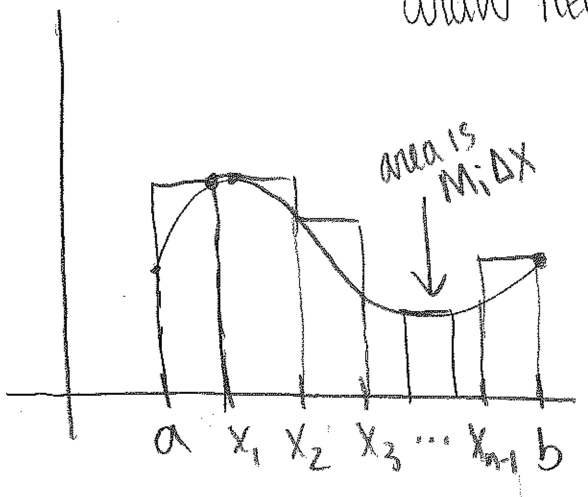
$\vdots$

$x_n = a + n \cdot \Delta x = b$

$\underbrace{P}_{\text{just a list telling us where to draw rectangles}} = \{x_0, x_1, x_2, \ldots, x_n\}$ (Partition)

# Upper Riemann Sum: (overestimate)

draw rectangles that cover all the area




Let $M_i$ be the max value of $f$ in an interval $[x_{i-1}, x_i]$
area of the corresponding rectangle is

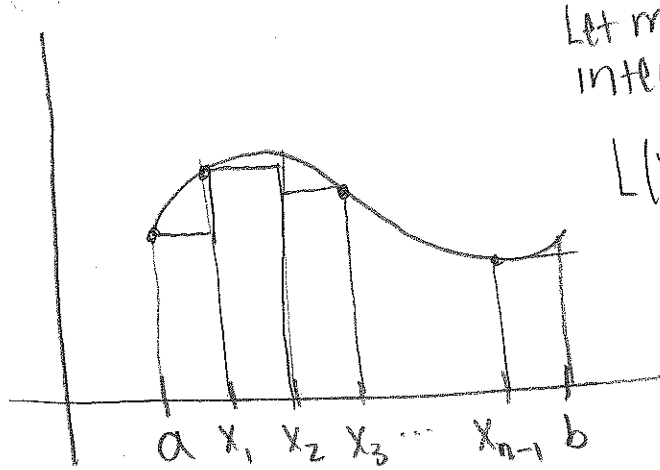
$M_i \cdot \Delta X$

Total area is $\underbrace{U}_{\text{upper}}(\underbrace{f}_{\text{func}}, \underbrace{P}_{\text{partition}}) = \underbrace{M_1}_{\text{height}} \underbrace{\Delta X}_{\text{width}} + M_2 \Delta X + M_3 \Delta X + \ldots + M_n \Delta X$

# Lower Riemann Sum: (underestimate)

Let $m_i$ be smallest value in interval $[x_{i-1}, x_i]$

$L(f, P) = m_1 \Delta X + m_2 \Delta X + \ldots + m_n \Delta X$



Idea: we will let # rectangles go to $\infty$.
($n \longrightarrow \infty$)

## Sigma Notation

A way to write addition more succinctly

- $f$ is a function
- $m, n$ are #'s w/ $m \leq n$

$$f(m)+f(m+1)+f(m+2)+\dots+f(n-1)+f(n)$$

$$= \sum_{\underbrace{i}_{\text{place holder}}=\underbrace{m}_{\text{lower bound}}}^{n \text{ upper bound}} \underbrace{f(i)}_{\text{what is being added}}$$

Ex: $\sum_{i=1}^{n} i = 1+2+3+\dots+n$

$\sum_{i=1}^{4} i = 1+2+3+4$

$\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+\dots+n^2$

So we can write

over-approximation
$$\begin{cases} U(f,P) = M_1\Delta X + M_2\Delta X + \ldots + M_n\Delta X \\ \quad = \sum_{i=1}^{n} M_i \Delta X \end{cases}$$

under-approx
$$\begin{cases} L(f,P) = m_1\Delta X + m_2\Delta X + \ldots + m_n\Delta X \\ \quad = \sum_{i=1}^{n} m_i \Delta X \end{cases}$$

<u>Defin of Area:</u>

Let A be area under $f(x)$ from a to b,

then $A = \lim_{n\to\infty} \sum_{i=1}^{n} m_i \Delta X$

<u>The Riemann Integral</u>

If there is a value $t$ s.t.

$U(f,P) > t > L(f,P)$

For any # of rectangles $n$ then

$$t = \int_a^b f(x)dx$$

$$\int_a^b f(x)\,dx = \text{area under } f(x) \text{ from } a \text{ to } b.$$

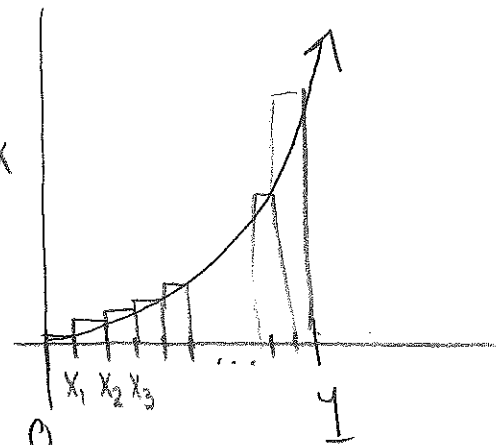
<u>Ex</u>: Back to $f(x) = x^2$ from $0$ to $1$.

Notice $x^2$ is increasing on this interval.

Approx w/ $n = 8$ rectangles:

$$U(f,P) = M_1\Delta x + M_2\Delta x + \dots + M_8\Delta x$$

$$= f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_8)\Delta x$$




$$\Delta x = \frac{b-a}{n} = \frac{1-0}{8} = \frac{1}{8}$$

$x_0 = 0$ $\quad x_3 = 3/8$

$x_1 = 1/8$ $\quad \vdots$

$x_2 = 1/4$ $\quad x_i = 0 + i\cdot\frac{1}{8} = i\cdot\frac{1}{8}$

max value

$M_i$ on $[x_{i-1}, x_i]$ is $f(x_i) = x_i^2$

$$= f\left(\tfrac{1}{8}\right)\left(\tfrac{1}{8}\right) + f\left(\tfrac{2}{8}\right)\left(\tfrac{1}{8}\right) + \dots + f\left(\tfrac{8}{8}\right)\left(\tfrac{1}{8}\right)$$

$$= \sum_{i=1}^{8} f\left(\frac{i}{8}\right)\frac{1}{8} = \sum_{i=1}^{8}\left(\frac{i}{8}\right)^2\frac{1}{8} = \left(\frac{1}{8}\right)^2\left(\frac{1}{8}\right) + \left(\frac{2}{8}\right)^2\left(\frac{1}{8}\right) + \dots + \left(\frac{8}{8}\right)^2\left(\frac{1}{8}\right)$$